

# ON THE GEVREY CONVERGENCE OF SOME CHARACTERISTIC CAUCHY PROBLEMS

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**ABSTRACT.** We prove that the solutions of an initial value problem of a linear partial differential equation of degree  $s$  is of Gevrey class  $s$  and in general not of class  $(s - 1)$ .

## INTRODUCTION

The linear Cauchy-Kovalevskaja theorem states that any analytic initial value problem of order 1

$$\partial_t u = \sum_{|j| \leq 1} f_j \partial_z^j u, \quad u(t = 0, \cdot) = u_0, \quad u = (u_1, \dots, u_m), \quad z = (z_1, \dots, z_n)$$

admits an analytic solution. In the formula, we used multi-index notations  $\partial_z^j = \partial_{z_1}^{j_1} \dots \partial_{z_n}^{j_n}$  and  $|j| = j_1 + \dots + j_n$ . As Kovalevskaja noticed in her thesis, for the *heat equation*  $\partial_t u = \partial_{zz} u$  with initial value  $u_0 = \frac{1}{1-z}$ , the solution is not analytic [18]. In 1904, in his thesis on Hilbert's nineteenth problem, Bernstein proved that the formal flow of parabolic equations has analytic coefficients [2] (see also Theorem 4). The same year, Holmgren proved that although non necessarily analytic, the solutions to the heat equation belong to some generalisation of analytic functions, now called *Gevrey class 2* [12]. In 1913, Gevrey defined *Gevrey classes*: a formal power series  $\sum_n a_n t^n \in \mathbb{C}[[z_1, \dots, z_n, t]]$ ,  $a_n \in \mathbb{C}[[z_1, \dots, z_n]]$  is called of *Gevrey class  $s$*  if the series  $\sum_n a_n \frac{t^n}{(n!)^{s-1}}$  is analytic. The functions of Gevrey class  $s$  form a differential ring and are asymptotic expansions of holomorphic functions [6, 7] (see also [1, 15]). The initial value problem for the heat equation is non-characteristic but of a very special kind: it admits a unique formal solution.

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THEOREM 1. *The solutions of any linear analytic initial value problem of order  $s$*

$$\partial_t u = \sum_{|j| \leq s} f_j \partial_z^j u, \quad u(t=0, \cdot) = u_0, \quad u = (u_1, \dots, u_m), \quad z = (z_1, \dots, z_n)$$

*are of Gevrey class  $s$ .*

In the non-linear case, I was not able to prove or disprove the Gevrey convergence. It is of course tempting to reduce the theorem to a fixed point theorem, as Nagumo did for  $s = 1$  [16].

This theorem implies in particular that the solutions to such linear characteristic Cauchy problems are asymptotic expansions of holomorphic functions (compare with [17]).

In [14], Łysik proved that the solutions of the *Korteweg-de Vries* equation  $\partial_t u = \partial_z^3 u + u \partial_z u$  are of Gevrey class 3 and that for the initial value  $u_0 = \frac{1}{1-z^2}$  the solution is not of Gevrey class 2.

Let us now give a theorem which gathers both the Kovalevskaja and the Łysik non-convergence results.

We consider a holomorphic function

$$f : J^s(\mathbb{C}^n, \mathbb{C}^m) \supset U \longrightarrow \mathbb{C}^m$$

where  $J^s(\cdot, \cdot)$  denotes the space of  $s$ -jets of holomorphic maps and  $U$  is an open neighbourhood. The non-linear initial value problem associated to  $f$  consists in finding a holomorphic function  $u$  such that:

$$j^s(\partial_t u) = f(j^s u), \quad u(t=0, \cdot) = u_0$$

where the initial condition  $u_0$  is such that its  $s$ -jet extension  $j^s u_0$  lies in  $U$ . In classical notations:

$$\partial_t u = f(z, u, \partial_z u, \dots), \quad u(t=0, \cdot) = u_0.$$

This notation refers to the fact that the ring of functions on  $J^s(\mathbb{C}^n, \mathbb{C}^m)$  has canonical coordinates  $y_0 = z$ ,  $y_1 = u(z)$ ,  $y_2 = \partial_z u(z)$ .

Recall that a formal power series  $u = \sum_n a_n x^n \in \mathbb{C}[[x]]$ ,  $x = (x_1, \dots, x_p)$  is *majorated* by another formal power series  $v = \sum_n b_n x^n \in \mathbb{R}_+[[x]]$ , denoted  $u \ll v$ , if  $|a_n| \leq b_n$  for all values of  $n$ .

THEOREM 2. *Consider a non-linear initial value problem*

$$\partial_t u = f(z, u, \partial_z u, \dots), \quad u(t=0, \cdot) = u_0$$

*of order  $s$  such that<sup>1</sup>  $f \gg 0$ ,  $u_0 \gg 0$ . If the holomorphy domain of  $u_0$  is bounded then the solution to this initial value problem is not of Gevrey class  $(s-1)$ .*

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<sup>1</sup>These estimates have to be understood with respect to the respective canonical coordinates.

## 1. THE FORMAL FLOW OF A VECTOR FIELD

**1.1. The Taylor formula.** We denote by  $\mathcal{L}(E, F)$  the vector space of continuous linear mappings between locally convex vector spaces  $E, F$  for the topology of uniform convergence on bounded subset [3].

Differential calculus in locally convex spaces is intricate therefore we shall restrict ourselves to holomorphic maps.

A map  $P : E \longrightarrow F$  is called a *degree  $n$  homogeneous polynomial* if there exists a linear mapping  $\tilde{P} : \prod_{i=1}^n E \longrightarrow F$  so that  $P(u) = \tilde{P}(u, \dots, u)$ . Let  $E, F$  be two complex complete locally convex vector spaces and let  $U$  be an open neighbourhood in  $E$ . A mapping  $f : E \supset U \longrightarrow F$ , between is called *holomorphic* if it satisfies the following two conditions

- (1) it is continuous,
- (2) for any linear mappings  $j : \mathbb{C} \longrightarrow E$ ,  $\pi : F \longrightarrow \mathbb{C}$  the map  $\pi \circ f \circ j$  is holomorphic.

The Gâteaux derivative of  $f$  at  $u$  in the direction  $\xi$ , if it exists, is defined by

$$Df(u)\xi := \lim_{t \rightarrow 0} \frac{f(u + t\xi) - f(u)}{t}.$$

This map can be iterated, we denote by  $D^n f(u)\xi$  the  $n$ -th Gâteaux derivative of  $f$  at  $u$  in the direction  $\xi$ . For instance

$$D^2 f(u)\xi := \lim_{t \rightarrow 0} \frac{Df(u + t\xi)\xi - Df(u)\xi}{t}.$$

**THEOREM 3.** *Let  $f : E \supset U \longrightarrow F$  be a holomorphic mapping. For any  $u \in U$*

- (1) *there exists a sequence of degree  $n$ -homogeneous polynomials  $P_n : E \longrightarrow F$ ,  $n \in \mathbb{Z}_{\geq 0}$  such that  $D^n f(u)\xi = P_n(\xi)$ ,*
- (2) *the map  $f$  is the sum of its Taylor series:  $f(u + \xi) = \sum_{n \geq 0} \frac{1}{n!} D^n f(u)\xi$ .*

This result is classical, we refer to [5], Chapter 1 and 2 and historical references therein for more details. Like for Banach spaces, the map

$$Df : U \mapsto L(E, F)$$

is called the *differential*. If the map  $f$  is linear then  $Df(u) = f$  at any point  $u \in U$ . Remark, that the standard notation for the Taylor formula in locally vector spaces when applied to the case of a finite dimensional vector space differs from the one in elementary calculus. For instance, for a holomorphic function  $f : \mathbb{C} \longrightarrow \mathbb{C}$ , we write the Taylor formula at the origin as

$$f(\xi) = \sum_{n \geq 0} D^n f(0)\xi$$

instead of  $f(\xi) = \sum_{n \geq 0} f^{(n)}(0)\xi^n$ .

**1.2. The non-linear Taylor formula.** We denote by  $\mathcal{H}(U, E)$  the space of holomorphic maps from  $U$  to  $E$ . An element  $X \in \mathcal{H}(U, E)$  defines a derivation

$$L_X : \mathcal{H}(U, E) \longrightarrow \mathcal{H}(U, E), \quad g \mapsto [x \mapsto Dg(x)X(x)].$$

If  $U = E = \mathbb{C}^n = \{x_1, \dots, x_n\}$  and  $X = (f_1, \dots, f_n) \in \mathcal{H}(\mathbb{C}^n, \mathbb{C}^n)$  then  $L_X$  is the Lie derivative along the vector field  $\sum_k f_k \partial_{x_k}$ .

We denote by  $U[[t]]$ , the set of formal power series with coefficient in  $U$ :

$$U[[t]] = \left\{ \sum_{k \geq 0} v_k t^k : v_k \in U \right\}.$$

The projection degree by degree induces a bijection of the set  $U[[t]]$  with the product  $\prod_{i \in \mathbb{N}} U$ . This identification endows  $U[[t]]$  of a topology. As a topological vector space  $U[[t]]$  is isomorphic to the topological tensor product  $U[[t]] \approx U \hat{\otimes}_{\mathbb{C}} \mathbb{C}[[t]]$  [9]. Similarly, we put  $E[[t]] \approx E \hat{\otimes}_{\mathbb{C}} \mathbb{C}[[t]] \approx \prod_{i \in \mathbb{N}} E$ .

Using the Taylor formula (Theorem 3), the derivation  $L_X$  extends to a derivation in  $\mathcal{H}(U[[t]], E[[t]])$ . For instance, we have

$$X(u + tv) = X(u) + tDX(u)v + \frac{t^2}{2}D^2X(u)v + \dots$$

**DEFINITION 1.1.** The (formal) *flow* of a vector field  $X \in \mathcal{H}(U, E)$  at  $u_0$  is an element  $u \in U[[t]]$  such that  $\partial_t u = X(u)$  and  $u(t = 0, \cdot) = u_0$ .

The Taylor formula implies that the formal flow is unique. Formal flows are frequently considered in mathematical physics [4, 8].

**THEOREM 4.** *The flow of the holomorphic vector field  $X \in \mathcal{H}(U, E)$  at a point  $u_0 \in U$ , is obtained by evaluating the map*

$$e^{tL_X} \text{Id} = \text{Id} + tX + \frac{t^2}{2}L_X X + \frac{t^3}{3!}L_X^2 X + \dots \in \mathcal{H}(U[[t]], E[[t]])$$

at  $u_0$ .

*Proof.* The maps  $\varphi_t = e^{tL_X}$  form a one-parameter subgroup. Therefore

$$\varphi^{t+\varepsilon}(u_0) - \varphi^t(u_0) = \varphi^\varepsilon(\varphi^t(u_0)) - \varphi^t(u_0) = (\varepsilon L_X \text{Id})(\varphi^t(u_0)) \pmod{\varepsilon}.$$

As  $\text{Id}$  is a linear mapping, we have  $L_X \text{Id} = X$ . Thus

$$(\varepsilon L_X \text{Id})(\varphi^t(u_0)) = \varepsilon X(\varphi^t(u_0)).$$

This proves the theorem. □

*Example 1.1.* For  $E = \mathbb{C}$ , it is customary to use the symbols  $x, x_0, v$  instead of  $u, u_0, X$ . The Lie derivative of a function  $f$  along the vector field  $v\partial_x$  is the product

$$L_v : \mathcal{H}(\mathbb{C}, \mathbb{C}) \longrightarrow \mathcal{H}(\mathbb{C}, \mathbb{C}), \quad x \mapsto f'(x)v(x).$$

Therefore the formula

$$u(t) = u_0 + tX(u_0) + \frac{t^2}{2!}L_X X(u_0) + \frac{t^3}{3!}L_X^2 X(u_0) + \dots$$

becomes

$$x(t) = x_0 + tv(x_0) + \frac{t^2}{2!}v(x_0)v'(x_0) + \frac{t^3}{3!}v(x_0)(vv')'(x_0) + \dots$$

For instance, the vector field  $v(x) = \partial_x$  in  $\mathbb{C}$  being identified with the constant function  $x \mapsto 1$ , we get the formula

$$x(t) = x_0 + t$$

which indeed integrates the differential equation  $\dot{x} = 1$ .

The vector field  $v(x) = x\partial_x$  in  $\mathbb{C}$  is identified with the linear function  $x \mapsto x$ . We get the formula

$$x(t) = x_0 + tx_0 + \frac{t^2}{2!}x_0 + \frac{t^3}{3!}x_0 + \dots = e^t x_0.$$

which indeed integrates the differential equation  $\dot{x} = x$ .

*Example 1.2.* The polynomial ring  $E = \mathbb{C}[z]$  has a locally convex topology: a subset is open if its intersection with any finite dimensional subvector space is open. The solution to the initial value problem

$$\partial_t u = \partial_z u, \quad u(t = 0, \cdot) = u_0$$

is given by  $(t, z) \mapsto u_0(t + z)$ . Let us now apply Theorem 4.

We consider the vector field

$$X : \mathbb{C}[z] \longrightarrow \mathbb{C}[z], \quad u \mapsto \frac{du}{dz}.$$

As  $X$  is linear, the differential of the function  $X$  at any point is equal to  $X$ , i.e.,

$$DX(u) = X, \quad \forall u \in \mathbb{C}[z].$$

Therefore the flow of  $X$  at  $u_0$  is

$$u(t, \cdot) = u_0 + t \frac{du_0}{dz} + \frac{t^2}{2!} \frac{d^2 u_0}{dz^2} + \frac{t^3}{3!} \frac{d^3 u_0}{dz^3} + \dots$$

In this case the formula for the flow reduces to the Taylor formula

$$u_0(z + t) = u_0(z) + t \frac{du_0}{dz}(z) + \frac{t^2}{2!} \frac{d^2 u_0}{dz^2}(z) + \frac{t^3}{3!} \frac{d^3 u_0}{dz^3}(z) + \dots$$

More generally if  $X$  is a linear mapping, then using the equality  $DX(u) = X$ , the formula for the flow reduces to

$$u = e^{tX} u_0 = \sum_n \frac{t^n}{n!} X^n(u_0).$$

Let us now consider the non-linear initial value problem:

$$\partial_t u = u \partial_z u, \quad u(t = 0, \cdot) = u_0$$

The solution of this initial value problem is obtained by taking the flow of the vector field

$$X : \mathbb{C}[z] \longrightarrow \mathbb{C}[z], \quad u \mapsto u \frac{du}{dz}$$

at  $u_0$ . At first order in  $t$ , the flow is given by evaluating  $\text{Id} + tL_X \text{Id}$  at  $u_0$ , we get

$$u(t, \cdot) = u_0 + tu_0 \frac{du_0}{dz} + \dots$$

To compute the second order term, we substitute  $\xi$  by  $X(u)$  in the formula

$$DX(u)\xi = \xi \frac{du}{dz} + u \frac{d\xi}{dz}.$$

We get that

$$L_X^2 \text{Id} = DX(u)X(u) = 2u \left( \frac{du}{dz} \right)^2 + u^2 \frac{d^2 u}{dz^2}.$$

and consequently the expansion for the flow at the second order in  $t$  is given by the formula

$$u(t, \cdot) = u_0 + tu_0 \frac{du_0}{dz} + \frac{t^2}{2!} \left( 2u_0 \left( \frac{du_0}{dz} \right)^2 + u_0^2 \frac{d^2 u_0}{dz^2} \right) + \dots$$

**1.3. Majorating vector fields.** The ring  $\mathcal{O}_n$  of germs of holomorphic map at  $0 \in \mathbb{C}^n$  form a complete locally convex topological vector space. The topology of this space is defined as follows [10]. Denote by  $D_r \subset \mathbb{C}^n$  the closed polydisk centred at the origin of polyradi  $(r, \dots, r)$ . Let  $E_r \subset \mathcal{O}_n$  be the subspace of functions which are continuous in  $D_r$  and holomorphic in the interior of  $D_r$ . The vector space  $E_r$  is a Banach space for the supremum norm:

$$\|u\|_r = \sup_{z \in D_r} |u(z)|.$$

A subset  $U \subset \mathcal{O}_n$  is *open* if its intersection with  $E_r$  is an open subset of  $E_r$  for any value of  $r > 0$ . This topology induces a topology on the product space  $\mathcal{O}_n^m = \prod_{i=1}^m \mathcal{O}_n$ .

Let  $X, Y$  be two vector fields in  $\mathcal{O}_n^m$ . A vector field  $X$  in  $\mathcal{O}_n^m$  is majorated by another one  $Y$  if  $u \ll v$  implies  $X(u) \ll Y(v)$ . The following proposition is a direct consequence of Theorem 4.

**PROPOSITION 1.1.** *Let  $X, Y$  be two vector fields defined in an open subset of  $\mathcal{O}_n^m$ .*

- (1) *If  $X \ll Y$  then the flow of  $X$  at  $u_0 \gg 0$  is majorated by that of  $Y$  at the same point,*
- (2) *If  $X \gg 0$  and  $v_0 \gg u_0$  then the flow of  $X$  at  $u_0$  is majorated by that of  $X$  at  $v_0$ .*

## 2. PROOF OF THE THEOREMS

### 2.1. Proof of Theorem 1.

**PROPOSITION 2.1.** *The following assertions are equivalent*

- (1) *the flow of any system of linear partial differential equations of order  $s$  in  $\mathcal{O}_n^m$  is of Gevrey class  $s$ ,*
- (2) *the flow of a linear partial differential equation of order  $s$  in  $\mathcal{O}_n$  is of Gevrey class  $s$ ,*
- (3) *the flow of a linear partial differential equation of order  $s$  at  $u_0 = \frac{1}{1-z}$  in  $\mathcal{O}_1$  is of Gevrey class  $s$ ,*
- (4) *the flow of  $\frac{1}{1-z} \partial_z^s$  at the point  $u_0 = \frac{1}{1-z}$  is of Gevrey class  $s$ .*

*Proof.* Consider a vector field

$$X : \mathcal{O}_n^m \supset U \longrightarrow \mathcal{O}_n^m, u \mapsto \sum_{|j| \leq s} f_j \partial_z^j u; j = (j_1, \dots, j_n), |j| = j_1 + \dots + j_n$$

at a point  $u_0$ .

I assert that it is sufficient to consider the case  $X \gg 0$ ,  $U \subset \{u \gg 0\}$ . Given an analytic series  $\alpha = \sum_n a_n z^n$  put  $\text{abs } \alpha = \sum_n |a_n| z^n$ . Consider the initial value problem where the  $f_j$ 's and  $u_0$  are replaced by  $\text{abs } f_j$  and  $\text{abs } u_0$ . By Proposition 1.1, if the solution of this new initial value problem is of Gevrey class  $s$  then  $X, u_0$  has the same property. This proves the assertion.

Let us consider the linear mapping

$$\psi : \mathcal{O}_n^m \longrightarrow \mathcal{O}_n, (u_1, \dots, u_m) \mapsto \sum_{k=1}^m u_k.$$

Write  $f_j = (f_{j1}, \dots, f_{jm}) \gg 0$  and put  $g_j = \sum_k f_{jk}$ . For any  $u \gg 0$ , we have

$$\psi \left( \sum_{|j| \leq s} f_j \partial_z^j u \right) = \sum_{k=1}^m \sum_{|j| \leq s} f_{jk} \partial_z^j u_k \ll \sum_{|j| \leq s} \left( \sum_{k=1}^m f_{jk} \right) \partial_z^j \left( \sum_{k=1}^m u_k \right) = \sum_{|j| \leq s} g_j \partial_z^j \psi(u).$$

Thus, by Theorem 4, the image under  $\psi$  of the flow of  $X$  at  $u_0$  is majorated by the flow of  $\sum_j g_j \partial_z^j$  at  $\psi(u_0)$ . This shows that (2)  $\implies$  (1).

Consider the open subset  $U = \{u \gg 0\} \subset \mathcal{O}_n$ . The mapping

$$R : \mathbb{C} \longrightarrow \mathbb{C}^n, \quad z \mapsto (z, \dots, z)$$

induces a map  $R^* : \mathcal{O}_n[[t]] \supset U[[t]] \longrightarrow \mathcal{O}_1[[t]]$ . An element is of Gevrey class  $s$  provided that its image under  $R^*$  is of Gevrey class  $s$ .

From the equalities  $R^* \partial_{z_i} z_j^k = k z^{k-1} \delta_{ij}$  and  $\frac{d}{dz} R^* z_j^k = k z^{k-1}$ , we get the estimate  $R^* \partial_{z_i} \ll \frac{d}{dz} R^*$ . Let  $X : u \mapsto \sum_j f_j \partial_z^j u$ ,  $f_j \gg 0$  be a vector field in  $\mathcal{O}_n$ . As  $R^* \partial_{z_i} \ll \partial_z R^*$ , the flow of the vector field

$$\mathcal{O}_1 \longrightarrow \mathcal{O}_1, \quad u \mapsto \sum_{|j| \leq s} R^* f_j \frac{d^{|j|} u}{dz^{|j|}}$$

at  $R^* u_0, u_0 \gg 0$  majorates the image under  $R^*$  of the flow of  $X$ . This shows that (3)  $\implies$  (2).

Let us now prove that (4)  $\implies$  (3).

For any open subset  $U \subset \mathcal{O}_1$  and any  $u_0 \in U$ , there exists a map of the type  $v_{A,B} = \frac{A}{B-z}$  contained in  $U$  which majorates  $u_0$ . If  $X \gg 0$ , by Theorem 4, the formal flow passing through  $u_0$  is majorated by the formal flow passing through  $v_{A,B}$ . Therefore it is sufficient to consider the case  $u_0 = v_{A,B}$  and up to multiplying  $u$  and  $z$  by constants, we may assume that  $u_0 = \frac{1}{1-z}$ . Take  $X = \sum_{j=0}^s f_j \frac{d^j}{dz^j}$  and  $u_0 = \frac{1}{1-z}$ . As  $\frac{d^s}{dz^s} u_0 \gg \frac{d^j}{dz^j} u_0$  for any  $j < s$ , we get that the flow of  $X$  at  $u_0$  is majorated by that of  $(\sum_{j=0}^s f_j) \frac{d^s}{dz^s}$ . The functions  $u_0, f(u_0)$  are majorated by some function  $v_{A,B}$ , and again without loss of generality we may assume that they are majorated by  $\frac{1}{1-z}$ . By Theorem 4, the flow of the vector field  $u \mapsto \frac{1}{1-z} \frac{d^s}{dz^s} u$  at  $u_0$  majorates the flow of  $(\sum_{j=0}^s f_j) \frac{d^s}{dz^s}$ . This proves that (4)  $\implies$  (3) and concludes the proof of the proposition.  $\square$

To conclude the proof of the theorem, it is sufficient to compute the flow of the vector field  $X = \frac{1}{1-z} \frac{d^s}{dz^s}$  at  $u_0 = \frac{1}{1-z}$ . By Theorem 4, we get that

$$u(t) = \sum_{j \geq 0} u_j \frac{t^j}{(1-z)^{js+j+1}}, \quad u_j = \frac{j((s+1)j-1)!}{(s+1)^{j-1}(j!)^2}.$$

The Stirling formula implies that there exists  $R \geq 1$  such that  $u_j \leq (j!)^{s-1} R^j$ . This concludes the proof of Theorem 1.

**2.2. Proof of Theorem 2.** As  $f \gg 0$  is of order  $s$  there exists  $j = (j_1, \dots, j_n)$  with  $|j| = s$  such that

$$f \gg g \partial_z^j, \quad g \gg 0.$$



Write  $u_0 = (u_{0,1}, \dots, u_{0,m})$  and assume that  $u_0 \gg 0$ . As the holomorphy domain of the initial condition  $u_0$  is compact there exists  $A, B \in \mathbb{R}_{>0}$ ,  $N \in \mathbb{Z}_{\geq 0}^n$ , such that the components  $u_{0,j}$  of  $u_0$  are such that

$$u_{0,j} \gg z^N \prod_{i=1}^n \frac{1}{(A - Bz_i)^{j_i}}, \quad \forall j.$$

Up to a multiplication of  $z$  and  $u_0$  by a constant, we may assume that  $A = B = 1$ .

Consider the linear vector field  $Y : u \mapsto g(j^s u_0) \partial_z^j u$ . For any  $v \ll u_0$ , we have  $Y^k u_0 \ll (g \partial_z^j)^k u_0$ , thus by Theorem 4, the flow of  $g \partial_z^j$  at  $u_0$  majorates that of  $Y$  at  $u_0$ .

As  $g \gg 0$  and  $u_0 \gg 0$ , the function

$$z \mapsto g(j^s u_0(z))$$

majorates any monomials in its Taylor expansion. It is therefore sufficient to prove the theorem for the initial value problem

$$\partial_t u = Lu, \quad u_{0,j} = z^N \prod_{i=1}^n \frac{1}{(1 - z_i)^{j_i}}$$

with  $L = z^\alpha \partial_z^j$ . We have the estimate

$$L^k u_{0,l} \gg z^{kN+k\alpha} \prod_{i \in I} (kj_i)! \frac{1}{(1 - z_k)^{kj_i+1}}$$

where  $I$  denotes the set of indices for which  $j_i \neq 0$ .

By Stirling's formula, there exists  $r > 0$  such that  $\prod_{i \in I} (kj_i)! \geq r^k (k!)^s$ . Theorem 4 implies that the flow of  $f$  at  $u_0$  majorates the map

$$\sum a_k(z) r^k t^k (k!)^{s-1}, \quad a_k(z) = z^{kN+k\alpha} \prod_{i \in I} \frac{1}{(1 - z_k)^{kj_i+1}}$$

which is not of Gevrey class  $(s-1)$ . This proves the theorem.

## APPENDIX A. THE KOVALEVSKAIA EXAMPLE

For a "generic" partial differential equation, the solutions are tangent along the characteristic and the order of tangency is arbitrary [11]. For evolution equations, we have seen that the situation is different: the formal solution is unique. The non-unicity of the Cauchy problem is given by an infinity of solution which differ from a small exponential. We illustrate this phenomenon in the Kovalevskai a example (see also [13])

$$\partial_t u = \partial_{zz} u, \quad u(t = 0, z) = \frac{1}{1 - z}.$$

According to Theorem 4, the solution to this initial value problem is given by the formal power series

$$u(t, z) = \sum_{j,k} (\partial_z^{2k} u_0) \frac{t^k}{k!} = \frac{1}{1 - z} \sum_{j,k} \frac{(2k)!}{k!} \left( \frac{t}{1 - z^2} \right)^k.$$

Remark that if we take  $z = 0$  we get the non-analytic expansion

$$u(t, 0) = \sum_k \frac{(2k)!}{k!} t^k \gg \sum_k k! t^k.$$

The expansion  $u$  is of Gevrey class 2, i.e., the expansion

$$\hat{u}(t, z) = \frac{1}{1 - z} \sum_{j,k} \frac{(2k)!}{(k!)^2} \left( \frac{t}{1 - z^2} \right)^k = \frac{1}{1 - z} \frac{1}{\sqrt{1 - 4 \frac{t}{1 - z^2}}}$$

is analytic. Put  $w = \frac{1 - z^2}{t}$ , then  $(1 - z)u - 1$  is the asymptotic expansion of the Laplace integral

$$f_\Gamma := \int_\Gamma \exp^{-\xi w} \frac{1}{\sqrt{1 - 4\xi}} d\xi$$

where  $\Gamma$  is any path in the half-plane  $H = \{\xi \in \mathbb{C} : \operatorname{Re} \xi > 0\}$  which avoids the singularity at  $\xi = 1/4$  and is asymptotic to a non-vertical half line. Thus, any function

$$u_\Gamma(z) = \frac{1}{1 - z} f_\Gamma(w) + 1, \quad w = \frac{1 - z^2}{t}$$

might be considered a solution to our initial value problem, i.e., it is a holomorphic solution to the heat equation and its asymptotic expansion at  $z = t = 0$  is the formal flow at  $u_0$ .

Let us now investigate, the dependence of the solution on the choice of

the path  $\Gamma$ . If we consider the half-lines  $L_{\pm} = \{\xi \in H : \operatorname{Re} \xi = \pm \operatorname{Im} \xi\}$ , we get two functions

$$f_{\pm} = \int_{L_{\pm}} \exp^{-\xi w} \frac{1}{\sqrt{1-4\xi}} d\xi.$$

By Cauchy's integral theorem, these two solutions differ by

$$a(w) = \int_{1/4}^{+\infty} \exp^{-\xi w} \frac{1}{\sqrt{1-4\xi}} d\xi$$

where the integral is taken for  $\xi \in \mathbb{R}$ . The analytic continuation of  $f_+$  when  $\xi$  turns around the origin is  $f_+ + a$ . Remark that the function  $a$  is flat: its asymptotic expansion as a series in powers of  $w^{-1}$  vanishes. This agrees with the fact that for any choice of  $\Gamma$  the asymptotic expansion of  $f_{\Gamma}$  induces a solution of our initial value problem.

By Cauchy's integral theorem, we have  $f_{\Gamma} = f_+ + ka$  where  $k$  is the index of  $\Gamma$  around the point  $\xi = 1/4$ . Thus, there are infinitely many solutions which differ from a small exponential and the fundamental group of  $\mathbb{C} \setminus \{0\}$  acts transitively on these solutions.

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